Indian Statistical Institute, Bangalore Centre
B.Math. (III Year) : 2011-2012
Semester II : Mid-Semestral Examination
Probability III (Stochastic Processes)

02.03.2012 Time: $2\frac{1}{2}$ hours. Maximum Marks : 80

Note: The paper carries 82 marks. Any score above 80 will be taken as 80. State clearly the results you are using in your answers.

1. (5+10 = 15 marks) The transition probability matrix of a Markov chain on the state space $\{1, 2, 3, 4, 5, 6\}$ is given by

$$\mathbf{P} \quad = \quad \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

(i) Show that the Markov chain is irreducible.

(ii) Find the period.

2. (6+9 = 15 marks) Let $\{X_n\}$ be a Markov chain on a countable state space S with transition probability matrix P. For $x, y \in S$, let $G_n(x, y) =$ expected number of visits to y during times $j = 1, 2, \cdots, n$ for the Markov chain starting at x.

(i) Show that $G_n(x, y) = \sum_{k=1}^{n} P_{xy}^{(k)}$.

(ii) Let $x \in S$ be a positive recurrent state; suppose that $y \in S$ is accessible from x. Show that y is also positive recurrent.

3. (5+10 = 15 marks) Consider an unrestricted birth-death Markov chain $\{Y_n\}$ on $\mathbb{Z}$ with transition probabilities $P_{i,i+1} = p_i, P_{i,i-1} = q_i, P_{ij} = 0, j \neq i-1, i+1$, for all $i \in \mathbb{Z}$. Suppose $p_0 = q_0 = \frac{1}{2}$, $p_i = p^+ > q^+ = q_i$ for all $i \geq 1$, $p_i = p^- < q^- = q_i$ for all $i \leq -1$, with $p^+ + q^+ = 1, p^- + q^- = 1, 0 < p^+ < 1, 0 < p^- < 1$.

(i) Show that $\{Y_n\}$ is irreducible.

(ii) What can you say about recurrence/ transience of $\{Y_n\}$?

4. (15+5 = 20 marks) Let $0 < p < 1$. Let $\{X_n\}$ be a Markov chain on $\{0, 1, 2, \cdots\}$ with transition probabilities given by $P_{i,i+1} = p, P_{i0} = 1 - p,\ i = 0, 1, 2, \cdots$ and $P_{ij} = 0$, otherwise.

   (i) Find a stationary probability distribution of $\{X_n\}$.

   (ii) Is stationary probability distribution of $\{X_n\}$ unique?

5. (17 marks) Let $\{X_n : n = 0, 1, 2, \cdots\}$ be a Markov chain on a finite state space S. Suppose there are only two irreducible closed sets C_1, C_2 for the Markov chain. For $i = 1, 2$, let $\mu^{(i)}$ be the unique stationary probability distribution for the Markov chain restricted to C_i. Let μ be a probability distribution on S. Show that μ is a stationary probability distribution for $\{X_n\}$ if and only if $\mu = a\mu^{(1)} + (1 - a)\mu^{(2)}$, for some $0 \leq a \leq 1$.